Team Theme 2006

Triangles, and Polygons, and Circles, oh Pi!!!!

Years ago mathematicians knew there was a relationship between the area of a circle and its radius, and the circumference of a circle and its diameter. The area (A) is proportional to the square of the radius ($A = Kr^2$) and the circumference (C) is proportional to the diameter ($C = kd$). The trouble was they couldn't find out exactly what the K and k (constants of proportionality) values were. The path, or yellow brick road, you are about to follow is similar to the path followed by Archimedes 2250 years ago to uncover the mystery of what K and k are.

Triangles

To get us started let's look at triangles. We are going to look at isosceles triangles which have two sides of the same length which in turn means there are two angles of the same size. Also, we are going to look at right triangles of The Pythagorean Theorem fame.

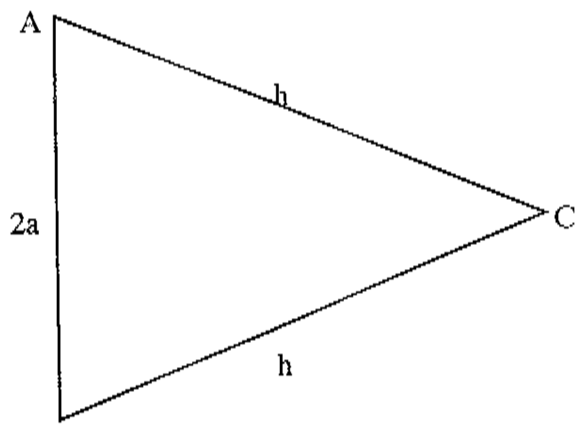


Fig. 1

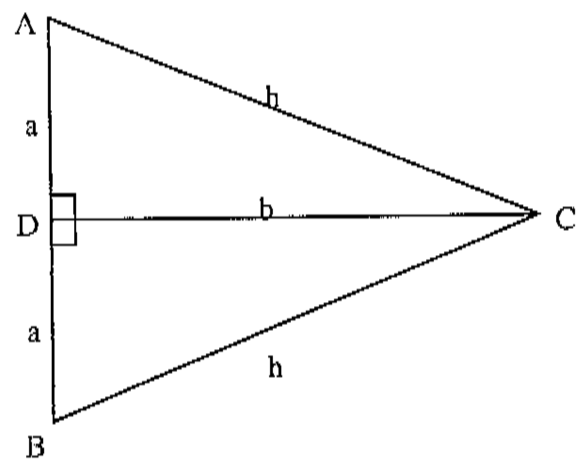
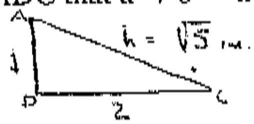


Fig. 2

Figure 1 is an isosceles triangle, while figure 2 is the same triangle with a perpendicular bisector drawn in. The perpendicular bisector $\overline{DC}$ intersects the side $\overline{AB}$ at the mid point D and forms a right angle with that side. For an isosceles triangle and for ours pictured, the perpendicular bisector intersects the vertex C forming the two similar right triangles.

Recall the area of a triangle like ΔADC is given by the formula $A = \frac{1}{2}ab$. Also recall the Pythagorean Theorem states for a triangle like ΔADC that $a^2 + b^2 = h^2$.

If b = 2", and a = 1".

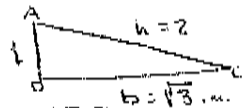


Area $\Delta ADC = \frac{1}{2} 2 \cdot 1 = 1$ sq. in

1. **Find the length of h**
2. **Find the area of the triangle ΔABC**

Area $\Delta ABC = 2$ sq. in.

If h = 2", and a = 1".



Area $\Delta ADC = \frac{1}{2} \sqrt{3} \cdot 1 = \frac{\sqrt{3}}{2}$ sq. in.

3. **Find the length of b**
4. **Find the area of the triangle ΔABC**

Area $\Delta ABC = \sqrt{3}$ sq. in.

Polygons

Polygons are many sided figures. We are going to concentrate our work on regular polygons, like squares (4 sides), hexagons (6 sides), octagons (8 sides) or n-gons (n sides). A regular polygon is a polygon whose sides are all the same length. Adjacent sides intersect at a point called the vertex. The angle formed at the vertex that is interior to the polygon is called a vertex angle (θ). The measure of each vertex angle in a regular polygon with n sides is given by the formula: $\theta = \frac{(n-2)180°}{n}$.

5. Find the measures of the vertex angles in each of the three polygons in Figure 3.

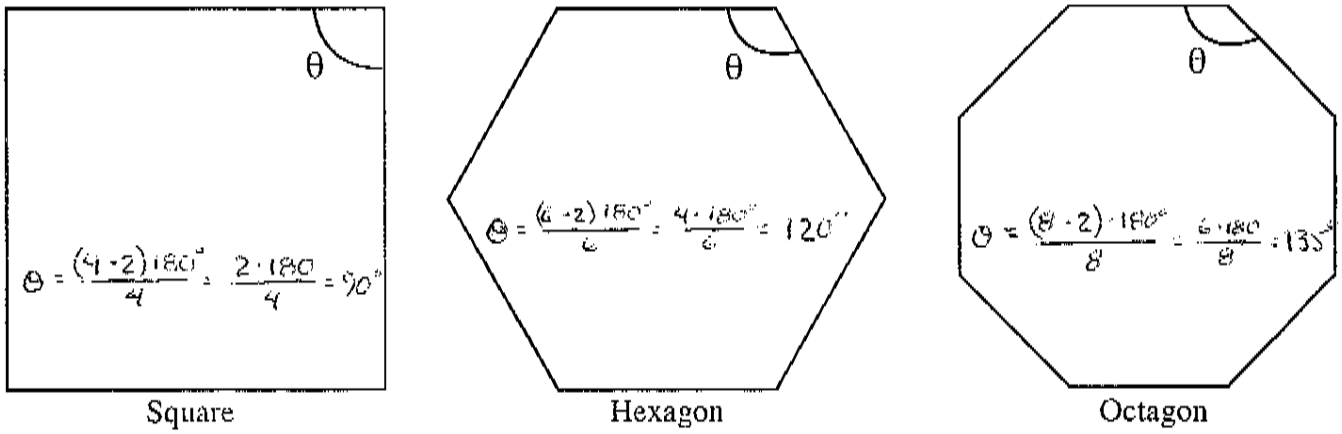


Fig. 3

A "center diagonal" of a polygon is a line segment drawn from one vertex to another passing through the center of the polygon. The angles formed between adjacent center diagonals are called central angles. The measure of the central angle in a regular polygon with n sides is given by the formula: $\beta = \frac{360°}{n}$. A triangle formed from the center and extending to the side of the polygon along two adjacent center diagonals is isosceles.

6. On the furnished paper with the pictures of 4 regular polygons, draw all of the "center diagonals". Find the measures of the central angles for each of the polygons.

If you take a regular polygon and fit it inside a circle such that the only points of contact are the vertices, then you have an inscribed polygon (see Fig. 4). If you take a circle and fit it inside a regular polygon such that the only points of contact are the midpoints of the sides, then you have a circumscribed polygon (see Fig. 5).

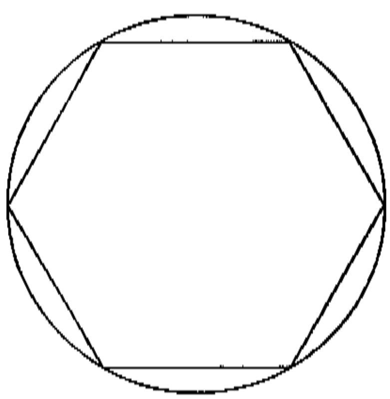


Fig. 4

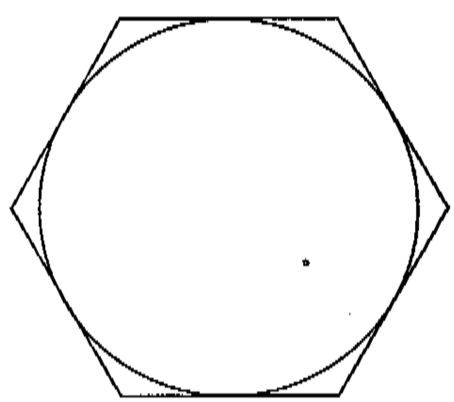


Fig. 5

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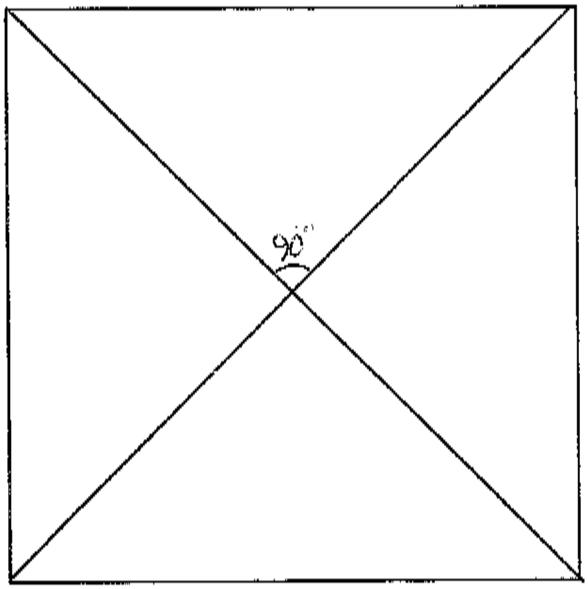
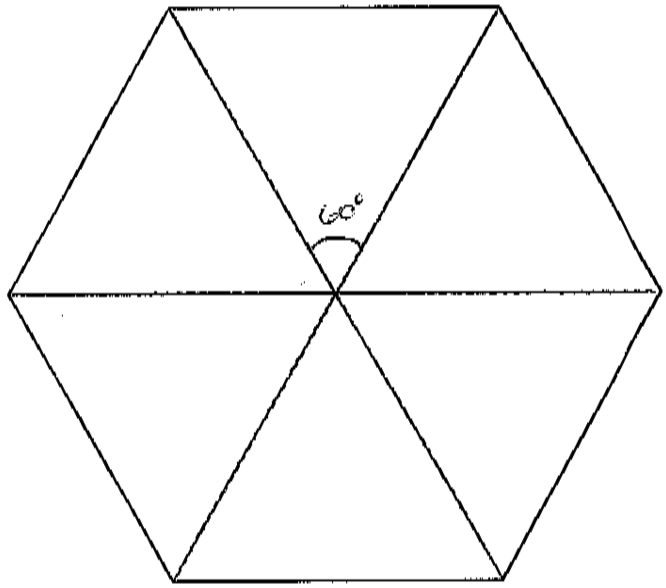
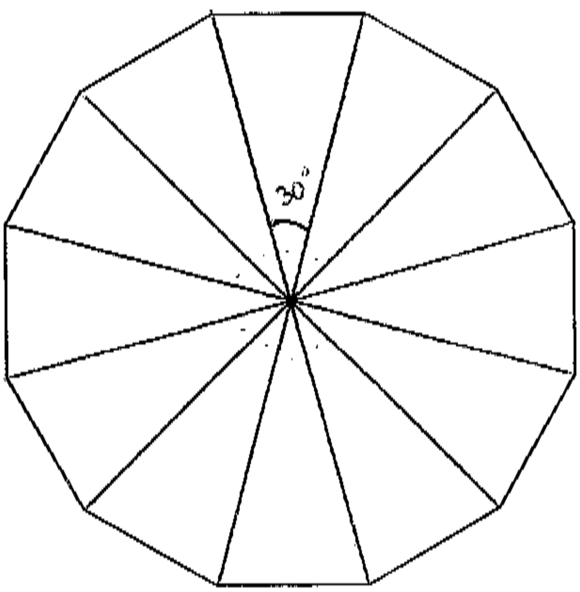
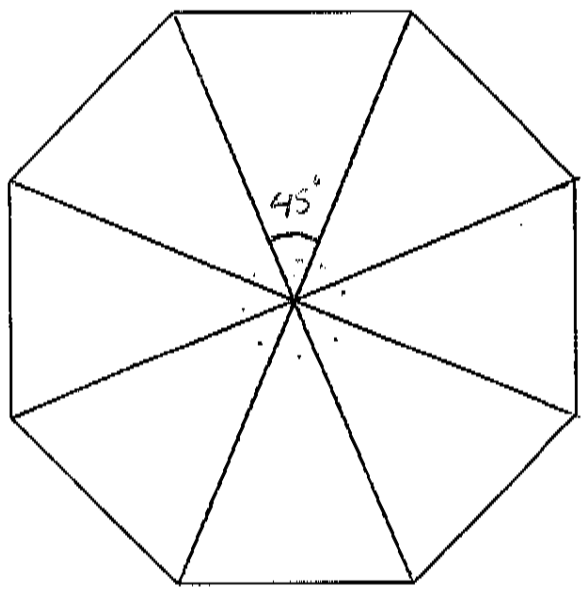
6. **On the furnished paper with the pictures of 4 regular polygons, draw all of the "center diagonals". Find the measures of the central angles for each of the polygons.**

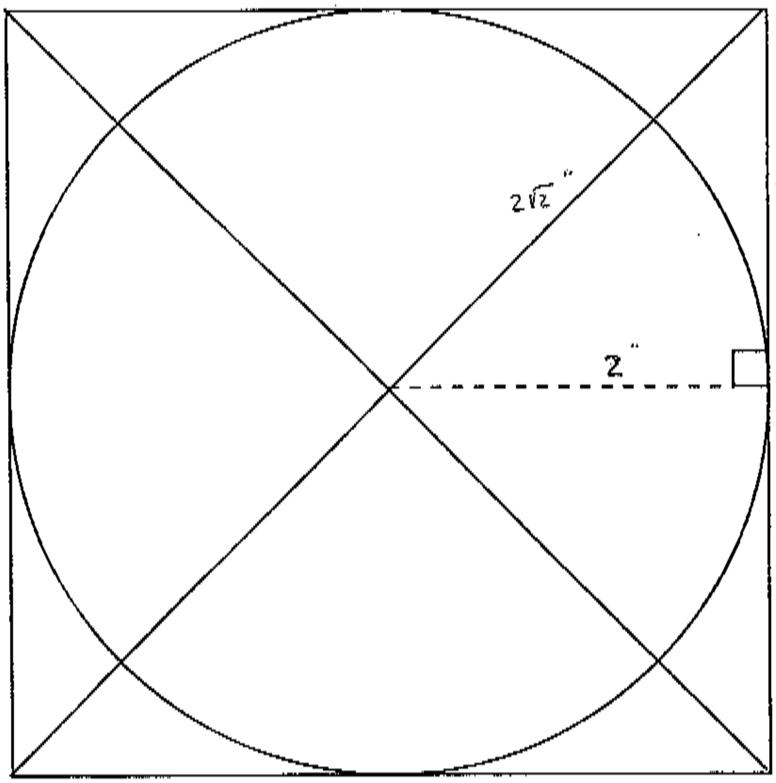








All circles have a radius of 2".



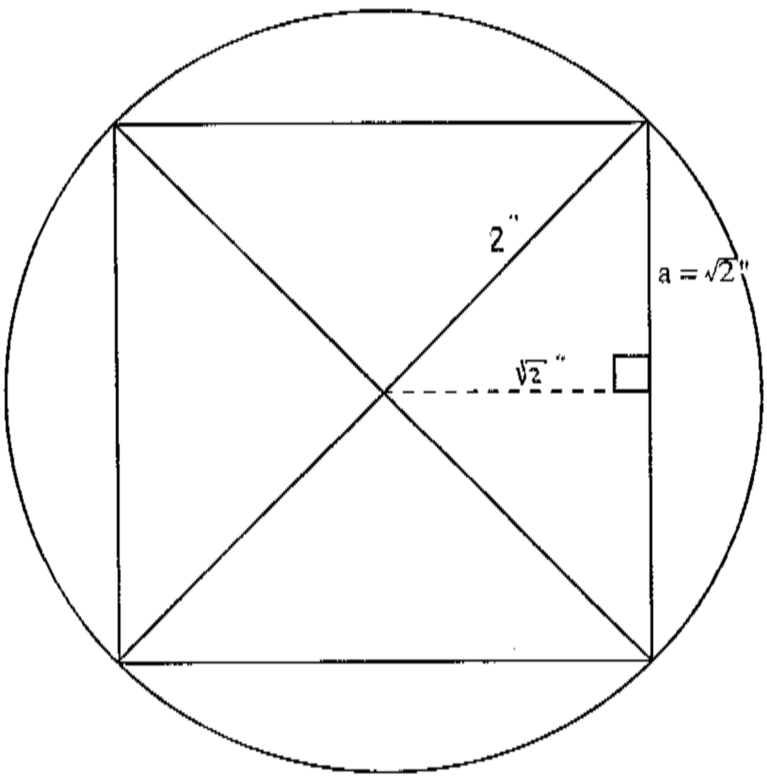
Area Right Tri = $\frac{1}{2}$ 2·2 = 2 sq. in.

Area Iso Tri = 4 sq. in.

Area of Circ Poly = 16 sq. in.

Length of Side 4 in.

Perimeter 16 in.



Area Right Tri = $\frac{1}{2}\sqrt{2}\cdot\sqrt{2}$ = 1 sq. in

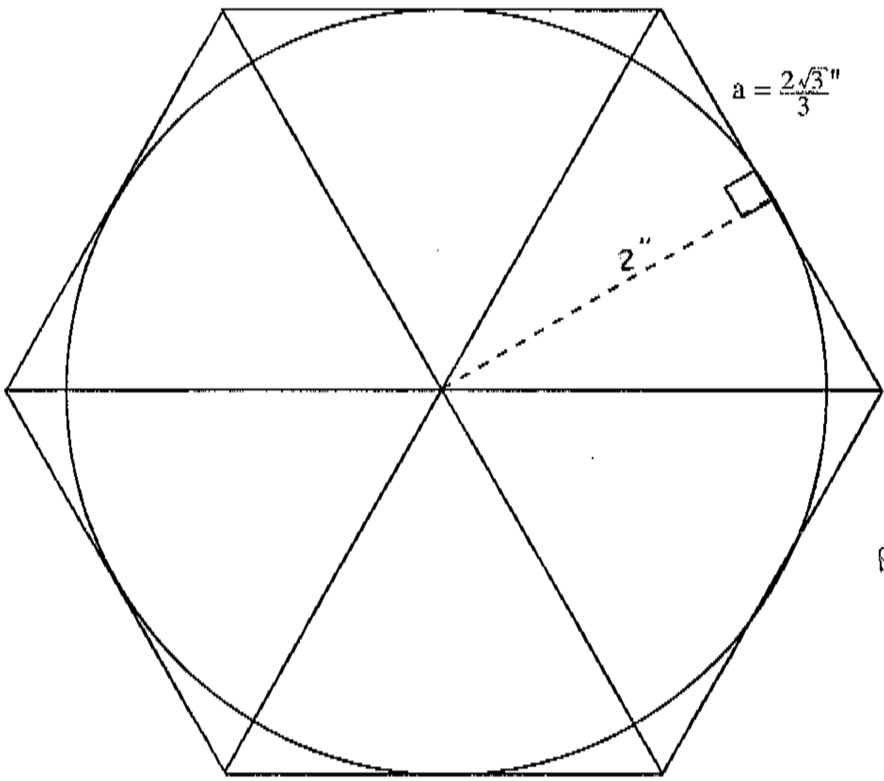
Area of Iso Tri = 2 sq. in.

Area of Insc. Poly = 8 sq. in.

Length of Side $2\sqrt{2}$ in.

Perimeter $8\sqrt{2}$ in.

All circles have a radius of 2".



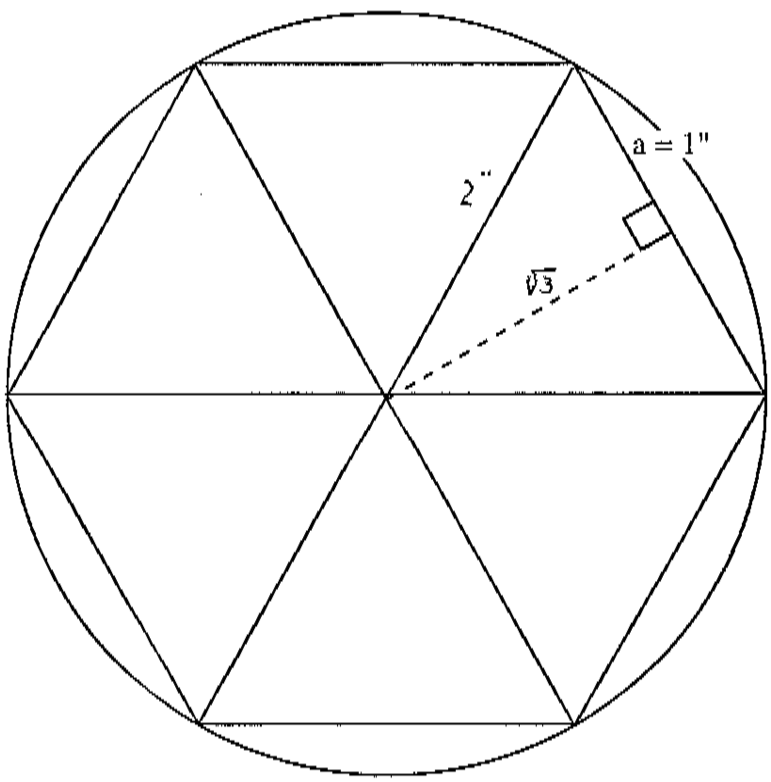
AREA OF RT. TR. $= \frac{1}{2} \cdot 2 \cdot \frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$ SQ

AREA OF ISO TR $= \frac{4\sqrt{3}}{3}$ SQ. IN.

AREA OF CIRC. POLY $= 8\sqrt{3}$ SQ. IN

LENGTH OF SIDE $\frac{4\sqrt{3}}{3}$ IN.

PERIMETER OF POLY $8\sqrt{3}$ IN.



AREA OF RT. TRI. $= \frac{1}{2} \, 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$ SQ. IN.

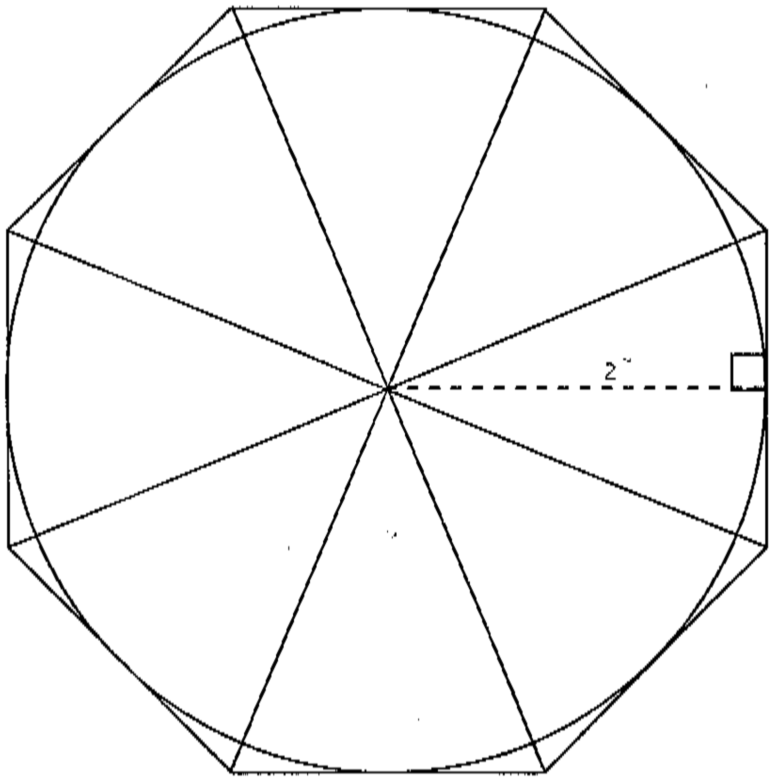
AREA OF ISO TRI. $= \sqrt{3}$ SQ. IN.

AREA OF INSC. POLY $= 6\sqrt{3}$ SQ. IN.

LENGTH OF SIDE 2 IN.

PERIMETER OF POLY. 12 IN.

All circles have a radius of 2".



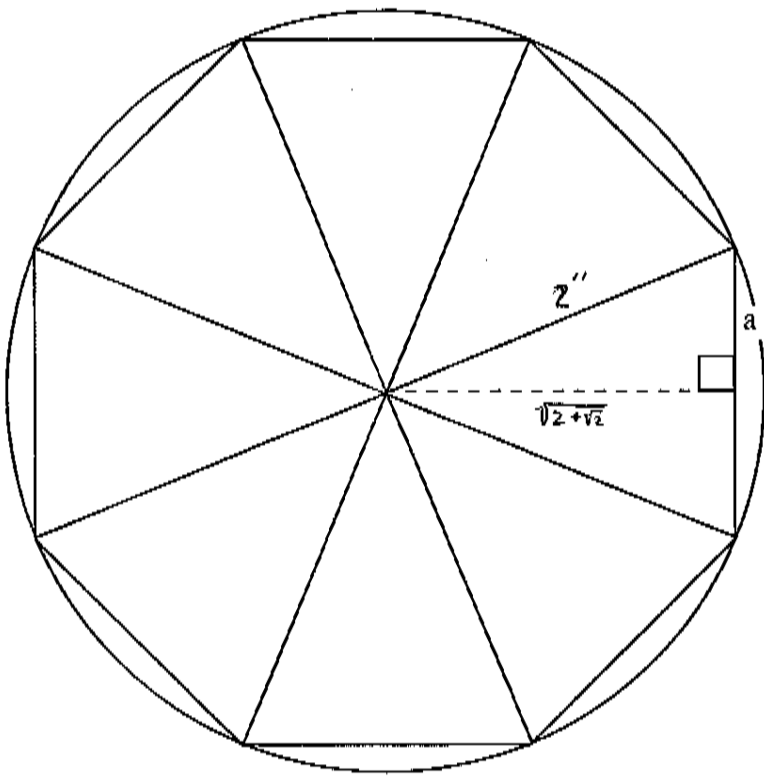
Area of Rt. Tr. $= \frac{1}{2}\, 2(2\sqrt{2}-2) = 2\sqrt{2} - 2$ sq.

Area of Iso. Tr. $= 4\sqrt{2} - 4$ sq. in.

Area of Poly $= 8(4\sqrt{2} - 4) = 32\sqrt{2} - 32$ s

Length of Side $4\sqrt{2} - 4$ in.

Per. of Poly $32\sqrt{2} - 32$ in.



Area of Rt. Tr. $= \frac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)\left(\sqrt{2-\sqrt{2}}\right) = \frac{\sqrt{2}}{2}$ sq. in.

Area of Is. Tr. $= \sqrt{2}$ sq. in.

Area of Poly. $= 8\sqrt{2}$ sq. in.

Length of Side $2\sqrt{2-\sqrt{2}}$ in.

Per. of Poly. $16\sqrt{2-\sqrt{2}}$ in.

The use of these inscribed and circumscribed polygons is how the mathematicians of old found the relationship between the area of a circle and its radius. They were able to approximate the value of K in the formula, $A = Kr^2$. The area of the circumscribed polygon is larger than the area of the circle while the area of the inscribed polygon is smaller than the area of the circle. The area of the circle will be squeezed in between the two polygon areas. The more sides the polygons have, the closer the areas will come to the actual area of the circle. The same can be said regarding the relationship between the perimeter of the polygon and the circumference of the circle. The perimeter of the circumscribed polygon is larger than the circumference of the circle while the perimeter of the inscribed polygon is smaller than the circumference of the circle. With more sides in the polygon the perimeter will get closer and closer to the actual circumference of the circle.

7. **In the formula $A = Kr^2$, where A is the area of the polygon, if $r = 2$ and $A = 10$, then what is K?**
   $10 = K \cdot 2^2$ $K = 2.5$
   $10 = K \cdot 4$

8. **In the formula $C = kd$, where C is the perimeter of the polygon, if $r = 2$ and $C = 10$, then what is k?**
   $10 = k \cdot 4$
   $2.5 = k$ $k = 2.5$

The sheets with the inscribed and circumscribed polygons are for your use in your attempt to approximate the values K and k in the formula from page 1: ($A = Kr^2$ and $C = kd$). Show all of your work on the lined paper provided.

9. **Enter all the values in the table. These entries are necessary to find the corresponding values of K and k. Turn in the table with the rest of your work.**

10. **What is the magic number that these K and k values are all approaching?**

11. **Archimedes is regarded as the greatest mathematician and scientist of antiquity and one of the three greatest mathematicians of all time.**

    a) **List some of Archimedes' accomplishments.** } vr mre → 3

    b) **List the other two greatest mathematicians of all time.**

    Newton
    Gauss

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n	Type	Area of right triangle	Area of isosceles triangle	Area of polynomial	K	Length of side of Polygon	Perimeter of Polygon	k
4	Inscribed	1 sq. in.	2 sq. in.	8 sq. in.	2	$2\sqrt{2}$ in. $\approx$ 2.828 in.	$8\sqrt{2}$ in. $\approx$ 11.3137 in.	2.828
4	Circum-scribed	2 sq	4 sq. in.	16 sq. in.	4	4 in.	16 in.	4
6	Inscribed	$\frac{\sqrt{3}}{2}$ sq. in. $\approx$ .866 sq. in.	$\sqrt{3}$ sq. in. $\approx$ 1.732 sq. in.	$6\sqrt{3}$ sq. in. $\approx$ 10.4 sq. in.	2.598	2 in.	12 in.	3
6	Circum-scribed	$\frac{2\sqrt{3}}{3}$ sq. in. $\approx$ 1.1547 sq. in.	$\frac{4\sqrt{3}}{3}$ sq. in. $\approx$ 2.3094 sq. in.	$8\sqrt{3}$ sq. in. $\approx$ 13.8564 sq. in.	3.464	$\frac{4\sqrt{3}}{3}$ in. $\approx$ 2.3094 in.	$8\sqrt{3}$ in. $\approx$ 13.8564 in.	3.464
8	Inscribed	$\frac{\sqrt{2}}{2}$ sq. in. $\approx$ .7071 sq. in.	$\sqrt{2}$ sq. in. $\approx$ 1.4142 sq. in.	$8\sqrt{2}$ sq. in. $\approx$ 11.3137 sq. in.	2.8284	$2\sqrt{2-\sqrt{2}}$ in. $\approx$ 1.5307 in.	$16\sqrt{2-\sqrt{2}}$ in. $\approx$ 12.2458 in.	3.0615
8	Circum-scribed	$2\sqrt{2}-2$ sq. in. $\approx$ .8284 sq. in.	$4\sqrt{2}-4$ sq. in. $\approx$ 1.6569 sq. in.	$32\sqrt{2}-32$ sq. in. $\approx$ 13.255 sq. in.	3.3137	$4\sqrt{2}-4$ in. $\approx$ 1.6569 in.	$32\sqrt{2}-32$ in. $\approx$ 13.255 sq. in.	3.3137

$A = Kr^2$

$8 = k \cdot 4$
$\frac{8}{4} = k$

$6\sqrt{3} = k \cdot 4$
$\frac{6\sqrt{3}}{4} = k$

$16 = k \cdot 4$
$\frac{16}{4} = k$

$\frac{8\sqrt{3}}{4} = k$

$P = k \cdot d$
$P = k \cdot 4$

$8\sqrt{2} = k \cdot 4$
$2\sqrt{2} = k$

$12 = k \cdot 4$
$3 = k$

$16 = k \cdot 4$
$4 = k$

$8\sqrt{3} = k \cdot 4$
$2\sqrt{3} = k$